

ANOTHER INCARNATION OF THE LAMBERT W FUNCTION

ALEXANDER KHEYFITS

To the memory of Jonathan M. Borwein (1951-2016)

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Abstract: *The Lambert W function was introduced by Euler in 1779, but was not well-known until it was implemented in Maple, and the seminal paper [1] was published in 1996. In this note we describe a simple problem, which can be straightforwardly solved in terms of the W function.*

At the recent seminar, Professor Bertram Kabak asked the following question:

The graphs of the exponential function e^x and of its inverse, the natural logarithm $\ln x$ have no point in common, see Fig. 1. Consider an exponential function with a base b , $0 < b$, $b \neq 1$,

$$y = b^x.$$

For what values of the base b , have the graphs of this function and its inverse, $y(x) = \log_b x$ the common points, and how many?

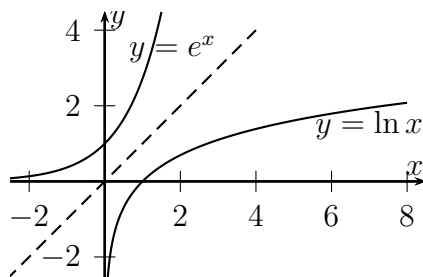


FIGURE 1. Graphs of exponential function $y = e^x$ and its inverse logarithmic function $y = \ln x$ never intersect.

It turns out that this problem can be easily solved by making use of the Lambert W function $W = w(z)$. The latter is given implicitly for every complex number z by

the equation

$$(1) \quad we^w = z,$$

where we follow the standard-now notation $w = w(z)$ [1].

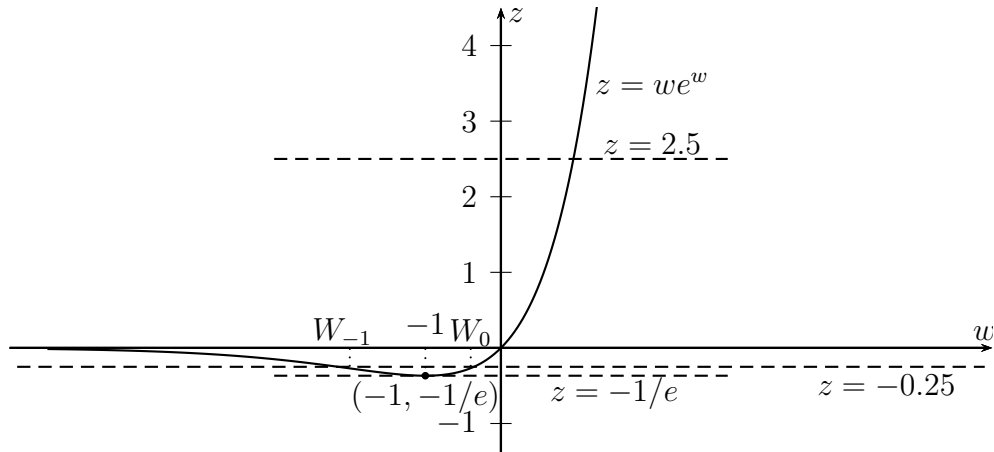


FIGURE 2. Function $z = we^w$. The dashed lines are the horizontal tangent line $z = -1/e$ and the horizontal lines $z = 2.5$ and $z = -0.25$.

After the W was implemented in Maple and other major CASs, and the article [1] has been published, the number of publications devoted to the function, has grown dramatically¹. The function enjoys so many applications, that together with the logarithmic function, the W should be in a toolbox of any researcher. Some authors claim that the W is an elementary function. Leave it to the individual judgment, whether and in what sense the Lambert W function is elementary, we show how the W naturally occurs in an elementary problem.

The Lambert W is a many-valued analytic function, therefore, its complete study and, in particular, description of its single-valued branches, should be done in complex-analytic terms, which is beyond the scope of this brief note, see, e.g., [1] or [3], where the closed-form representation of all the branches of W in terms of contour integrals was derived.

We mention only few necessary properties of the W function, referring the reader to the papers above. Begin by graphing the left-hand side of (1), see Fig. 2. The function $f(w) = we^w$ has the global minimum $-1/e$ at $w = -1$. For every $z \geq 0$, equation $z = we^w$ has exactly one real root, that is, the Lambert W function has one

¹On Jan. 8, 2017, the query 'Lambert W function' returned 302 references on MathSciNet, about 450,000 references on Google, and more than 1,200,000 references on Google Scholar.

real-valued branch. If $-1/e < z < 0$, equation (1) has two real roots, $-1 < W_0 < 0$ and $W_{-1} < -1$. When $z = -1/e$, these two roots merge in the double root $w = -1$. When $z < -1/e$, these roots disappear from the graph, because they became complex numbers.

The function $f(w) = we^w - z$ is an entire function for every complex z ; such functions are called quasi-polynomials. It is distinct from e^w , therefore, for any complex $z \neq 0$ this function has infinitely many roots. Since every root generates its own single-valued branch, the inverse function, Lambert W has infinitely many branches, and only two of them are real-valued on the real axis. They are conventionally called the principal branch, $W_0(z)$, which is real for $z \geq -1/e$, and the branch $W_{-1}(z)$, real-valued for $-1/e \leq z < 0$.

The graph of $W_0(z)$ is the mirror reflection, with respect to the bisectrix $w = z$, of the right half, $w \geq -1$, $z \geq -1/e$, of the graph in Fig. 2. The graph of $W_{-1}(z)$ is the mirror reflection of the left half of the same graph, $w \leq -1$, $-1/e \leq z \leq 0$, with respect to the bisectrix $z = w$.

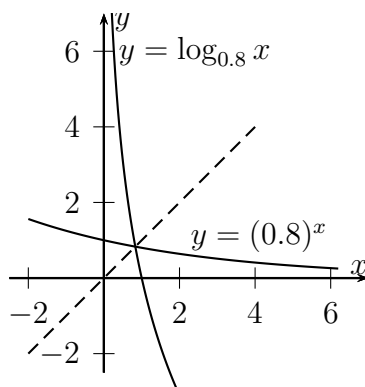


FIGURE 3. The functions $y = (0.8)^x$ and $y = \log_{0.8} x$ have the unique intersection point. The dashed line is the bisectrix $y = x$.

Now we can take up the question above. First, let be $0 < b < 1$. The graphs of a function and its inverse are symmetrical with respect to the bisectrix $z = w$, in the notations of Fig. 2, or the bisectrix $y = x$ in the notations of Fig. 3. Therefore, the intersection points, if any, must belong to the bisectrix, that is, satisfy the equation $b^w = w$, or $(-\ln b)we^{-(\ln b)w} = -\ln b$. The right hand side here is positive because $b < 1$, hence from Fig. 2 it follows that for any $0 < b < 1$ the equation has exactly one solution, or there exists the unique intersection point of b^w and $\log_b w$. It is worth repeating that the intersection point is given by the principal branch $W_0(\ln(1/b))$ of the Lambert W function. For instance, the case $b = 0.8$ is shown in Fig. 3, where the exponential and logarithmic curves intersect at the point

$W_0(-\ln 0.8)/(-\ln 0.8) \approx 0.83$. If we depart from the equation $w = \log_b w$, we arrive at the same conclusion.

The case $b > 1$ is a bit more interesting. The same reasoning leads to the equation $x = b^x$, or to equation (1),

$$we^w = z,$$

where we are to set $w = -x \ln b$ and $z = -\ln b < 0$. Since now $b > 1$ and $x > 0$, then $w = -x \ln b < 0$. Therefore, if $-1/e < z = -\ln b < 0$, that is, if $1 < b < e^{1/e}$, then there are two intersections, $W_0(z)$ and $W_{-1}(z)$, given by the principal branch W_0 and the preceding branch W_{-1} of the W function, see Fig. 4.

If $-1/e = z$, where $z = -\ln b$, that is, $b = e^{1/e}$, then the two points merge into the point of tangency, see Fig. 5. We can state the conclusion as follows.

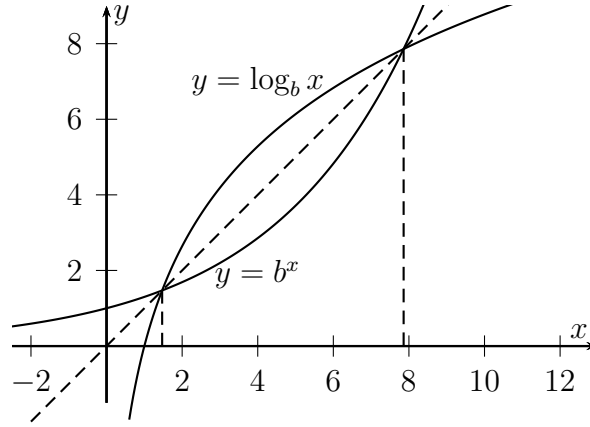


FIGURE 4. The functions $y = b^x$ and $y = \log_b x$ with $b = 1.3$ have two points of intersection.

Proposition 1. *For $0 < b < 1$, the graph of the exponential function $y = b^x$ and the graph of its inverse $y = \log_b x$ have the unique point of intersection, given by the principal branch of the Lambert W function $W_0(-\ln b)/(-\ln b)$. The graphs of the exponential function $y = b^x$ and of its inverse $y = \log_b x$ have a point of tangency if and only if $b = e^{1/e}$. The tangency point is x_t with $-x_t \ln b = -1$, that is, $x_t = e$. These graphs intersect exactly twice if and only if $1 < b < e^{1/e}$. The two points of intersection are given by the two branches, $W_0(-1/\ln b)$ and $W_{-1}(-1/\ln b)$. For example, if $b = 1.3$, the intersection points are (Fig.4)*

$$-W_0(-1/\ln 1.3)/\ln 1.3 \approx 0.386/\ln 1.3 \approx 1.47$$

and

$$-W_{-1}(-1/\ln 1.3) \approx 2.061/\ln 1.3 \approx 7.86.$$

For $b > e^{1/e}$, these graphs do not intersect (Fig. 1). □

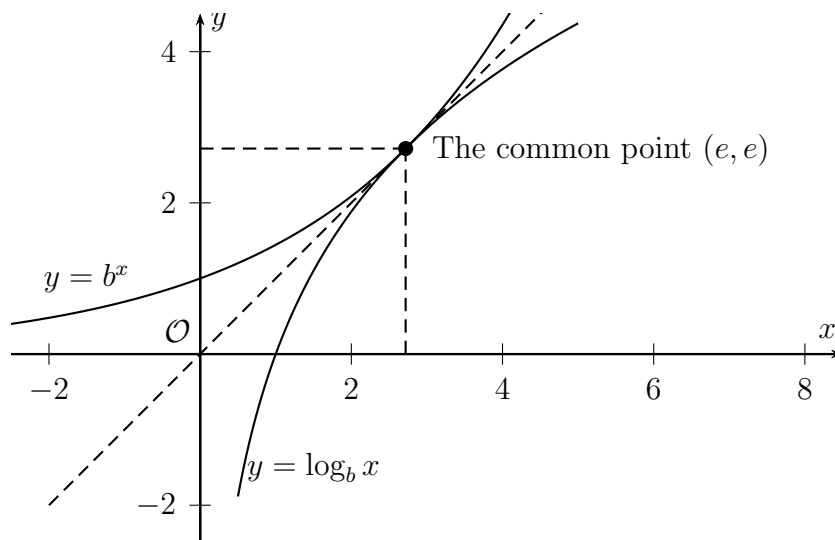


FIGURE 5. The functions $y = b^x$ and $y = \log_b x$ with $b = e^{1/e} \approx 1.44$ have a point of tangency.

The discussion leads to the following natural problem.

Problem. Describe those continuous increasing functions $f(x)$, $0 < x < \infty$, whose graphs have a) one, b) two, c) none, d) $n \geq 3$ points of intersection with the inverse function f^{-1} . The additional assumption of convexity will, definitely, simplify the problem and likely, lead to the functions $e^{f(x)}$.

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